

# **GLOBALLY IRREDUCIBLE WEYL MODULES**

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*Dedicated to Benedict Gross*

**ABSTRACT.** In the representation theory of split reductive algebraic groups, it is well known that every Weyl module with minuscule highest weight is irreducible over every field. Also, the adjoint representation of  $E_8$  is irreducible over every field. In this paper, we prove a converse to these statements, as conjectured by Gross: if a Weyl module is irreducible over every field, it must be either one of these, or trivially constructed from one of these.

## 1. INTRODUCTION

Split semisimple linear algebraic groups over arbitrary fields can be viewed as a generalization of semisimple Lie algebras over the complex numbers, or even compact real Lie groups. As with Lie algebras, such algebraic groups are classified up to isogeny by their root system. Moreover, the set of irreducible representations of such a group is in bijection with the cone of dominant weights for the root system and the representation ring — i.e.,  $K_0$  of the category of finite-dimensional representations — is a polynomial ring with generators corresponding to a basis of the cone.

One way in which this analogy breaks down is that, for an algebraic group  $G$  over a field  $k$  of *prime* characteristic, in addition to the irreducible representation  $L(\lambda)$  corresponding to a dominant weight  $\lambda$ , there are three other representations naturally associated with  $\lambda$ , namely the standard module  $H^0(\lambda)$ , the Weyl module  $V(\lambda)$ , and the tilting module  $T(\lambda)$ .<sup>1</sup> The definition of  $H^0(\lambda)$  is particularly simple: view  $k$  as a one-dimensional representation of a Borel subgroup  $B$  of  $G$  where  $B$  acts via the character  $\lambda$ , then define  $H^0(\lambda) := \text{ind}_B^G \lambda$  to be the induced  $G$ -module. The *Weyl module*  $V(\lambda)$  is the dual of  $H^0(-w_0\lambda)$  for  $w_0$  the longest element of the Weyl group and has head  $L(\lambda)$ . Typical examples of Weyl modules are  $\text{Lie}(G)$  for  $G$  semisimple simply connected ( $V(\lambda)$  for  $\lambda$  the highest root) and the natural module of  $\text{SO}_n$ . See [Jan03] for general background on these three families of representations.

It turns out that if any two of the four representations  $L(\lambda)$ ,  $H^0(\lambda)$ ,  $V(\lambda)$ ,  $T(\lambda)$  are isomorphic over a given field  $k$ , then all four are. Our focus is on the question: for which  $\lambda$  are all four isomorphic for *every* field  $k$ ? Consequently, it suffices to consider whether the Weyl module  $V(\lambda)$  — which for any  $k$  is obtained by base change from  $\mathbb{Z}$

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<sup>1</sup>The definitions of these three modules make sense also when  $\text{char } k = 0$ , and in that case all four modules are isomorphic.

— equals the irreducible module  $L(\lambda)$ . To emphasize this, we write  $V(\lambda)$  for the  $\mathbb{Z}$ -form and  $V(\lambda) \otimes k$  for the base change to  $k$ .

Whether  $V(\lambda) \otimes k$  is irreducible depends only on the characteristic of the field  $k$ . (Consequently, we are asking for which  $\lambda$  the module  $V(\lambda) \otimes \mathbb{F}_p$  is irreducible for all  $p$ . Such modules might be called *globally irreducible*.) There is a well known and elementary sufficient criterion:

(1) *If  $\lambda$  is minuscule, then  $V(\lambda) \otimes k$  is irreducible for every field  $k$ .*

(See §2 for the definition of minuscule.) This provides an important family of examples, because representations occurring in this way include  $\Lambda^r(V)$  for  $1 \leq r < n$  where  $V$  is the natural module for  $\mathrm{SL}_n$ ; the natural modules for  $\mathrm{SO}_{2n}$ ,  $\mathrm{Sp}_{2n}$ ,  $E_6$  and  $E_7$ ; and the (half) spin representations of  $\mathrm{Spin}_n$ .

While these representations play an outsized role, it is nevertheless true that in any reasonable sense they are a set of measure zero among the list of possible  $L(\lambda)$ . Therefore, we were surprised when Benedict Gross proposed to us that the sufficient condition (1) is quite close to also being a *necessary* condition, i.e., that there is only one other example. The purpose of this paper is to prove his claim, which is the following theorem.

**Theorem 1.1.** *Let  $G$  be a split, simple, and simply connected algebraic group over  $\mathbb{Z}$  and fix a dominant weight  $\lambda$  of  $G$ . In the following cases,  $V(\lambda) \otimes k$  is irreducible for every field  $k$ :*

- (a)  $\lambda$  is a minuscule dominant weight, or
- (b)  $G$  is a group of type  $E_8$  and  $\lambda$  is the highest root (i.e.,  $V(\lambda)$  is the adjoint representation for  $E_8$ );

*Otherwise, there is a prime  $p \leq 2(\mathrm{rank} G) + 1$  such that  $V(\lambda) \otimes k$  is a reducible representation of  $G$  for every field  $k$  of characteristic  $p$ .*

The bound  $2(\mathrm{rank} G) + 1$  is sharp by Theorem 5.1 below. The case where  $G$  is simple and simply connected (as in Theorem 1.1) is the main case. We have stated the theorem with these simplified hypotheses for the sake of clarity. See §2 for a discussion of the more general version where  $G$  is assumed merely to be reductive.

One surprising feature of our proof is the method we use to address a particular Weyl module of type  $B$  in §5, which we settle by appealing to modular representation theory of finite groups.

The literature contains some results complementary to Theorem 1.1, although we do not use them in our proof. For  $G$  of type  $A$ , Jantzen gave in [Jan03, II.8.21] a necessary and sufficient condition for the Weyl module  $V(\lambda)$  to be irreducible over fields of characteristic  $p$ . McNinch [McN98] (extending Jantzen [Jan96]) showed that for simple  $G$  and for  $\dim V(\lambda) \leq (\mathrm{char} k) \cdot (\mathrm{rank} G)$ ,  $V(\lambda)$  is irreducible.

We remark that John Thompson asked in [Tho76] an analogous question where  $G$  is finite: for which  $\mathbb{Z}[G]$ -lattices  $L$  is  $L/pL$  irreducible for every prime  $p$ ? This was extended by Gross to the notion of globally irreducible representations, see [Gro90] and [Tie97]. Our results demonstrate that  $F_4$  and  $G_2$  are the only groups that do not admit globally irreducible representations other than the trivial representation.

**Quasi-minuscule representations.** The representations appearing in (a) and (b) of Theorem 1.1 are *quasi-minuscule* (called “basic” in [Mat69]), meaning that the non-zero

weights are a single orbit under the Weyl group. For  $G$  simple, the quasi-minuscule Weyl modules are the  $V(\lambda)$  with  $\lambda$  minuscule or equal to the highest short root  $\alpha_0$ .

It is not hard to see that  $V(\alpha_0) \otimes k$  is reducible for some  $k$  when  $G$  is not of type  $E_8$ . If  $G$  has type  $A$ ,  $D$ ,  $E_6$ , or  $E_7$ , then  $V(\alpha_0)$  is the action of  $G$  on the Lie algebra of its simply connected cover  $\tilde{G}$ , and the Lie algebra of the center  $Z$  of  $\tilde{G}$  is a nonzero invariant submodule when  $\text{char } k$  divides the exponent of  $Z$ . The case where  $G$  has type  $B$  or  $C$  is discussed in §4. If  $G$  has type  $G_2$  or  $F_4$ , then  $V(\alpha_0)$  is the space of trace zero elements in an octonion or Albert algebra, and the identity element generates an invariant subspace if  $\text{char } k = 2$  or  $3$  respectively.

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## 2. DEFINITIONS AND NOTATION

We will follow the notation and conventions presented in [Jan03]. Let  $G$  be a simple simply connected algebraic group,  $T$  be a maximal split torus of  $G$  and  $\Phi$  be the root system associated to  $(G, T)$ . Fix a choice of simple roots  $\Delta$ . Let  $B$  be a Borel subgroup containing  $T$  corresponding to the negative roots and let  $U$  denote the unipotent radical of  $B$ .

One can naturally view  $\Phi$  as contained in a Euclidean space  $\mathbb{E}$  with inner product  $\langle \cdot, \cdot \rangle$ . Let  $X(T)$  be the integral weight lattice obtained from  $\Phi$ . The set  $X(T)$  has a partial ordering defined as follows. If  $\lambda, \mu \in X(T)$ , then  $\lambda \geq \mu$  if and only if  $\lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$ .

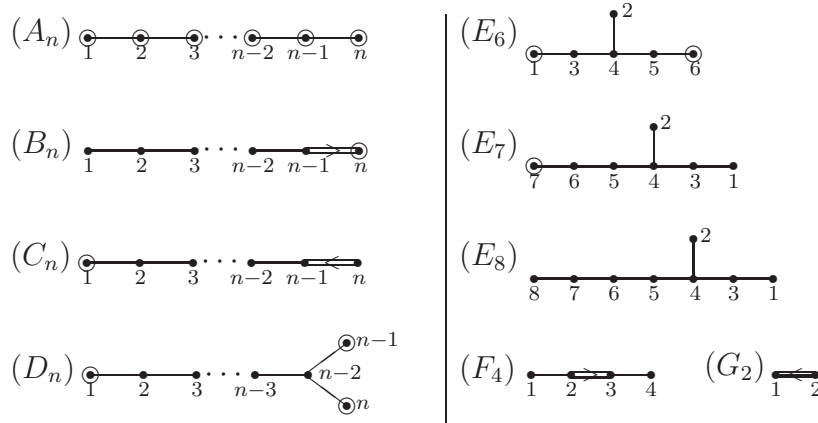


TABLE 1. Dynkin diagrams of simple root systems, with simple roots numbered. A circle around vertex  $i$  indicates that the fundamental weight  $\omega_i$  is minuscule.

For  $\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$  the coroot corresponding to  $\alpha \in \Phi$ , the set of dominant integral weights is defined by

$$X(T)_+ := \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Delta\}.$$

The fundamental weights  $\omega_j$  for  $j = 1, 2, \dots, n$  are the dual basis to the simple coroots. That is, if  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$  then  $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ .

We call the weights in  $X(T)_+$  that are minimal with respect to the partial ordering *minuscule* weights. Note that the zero weight is minuscule by this definition (in some references this is not the case). Every nonzero minuscule weight is a fundamental dominant weight (one of the  $\omega_i$ 's), and we have marked them in Table 1. We remark that there is a unique minuscule weight in each coset of the root lattice  $\mathbb{Z}\Phi$  in the weight lattice  $X(T)$  by [Bou02, §VI.2, Exercise 5a] or [Hum80, §13, Exercise 13]; this can be an aid for remembering the number of minuscule weights for each type and for determining which minuscule weight lies below a given dominant weight.

**Generalization of Theorem 1.1 to split reductive groups.** Suppose now that  $G$  is a split reductive group over a field  $k$ . Then there is a unique split reductive group scheme over  $\mathbb{Z}$  whose base change to  $k$  is  $G$ , which we denote also by  $G$ ; it is the split reductive group scheme over  $\mathbb{Z}$  with the same root datum as  $G$ . Moreover, there is a split reductive group scheme  $G'$  over  $\mathbb{Z}$  with a central isogeny  $G' \rightarrow G$  where  $G' = \prod_{i=0}^r G_i$  for  $G_0$  a torus and  $G_i$  simple and simply connected for  $i \neq 0$ , cf. [DG11, XXI.6.5.10]. A Weyl module  $V(\lambda)$  for  $G$  restricts to a Weyl module  $V(\sum \lambda_i)$  for  $G'$ , where  $\lambda_i$  denotes the restriction of  $\lambda$  to a maximal torus in  $G_i$ , and as in [Jan03, Lemma I.3.8] we have  $V(\sum \lambda_i) \cong \otimes_{i=0}^r V(\lambda_i)$ . Therefore,  $V(\lambda) \otimes k$  is an irreducible  $G$ -module for every field  $k$  if and only if  $V(\lambda_i) \otimes k$  is an irreducible  $G_i$ -module for every  $k$ , i.e., if and only if  $(G_i, V(\lambda_i))$  satisfies condition (a) or (b) of Theorem 1.1 for all  $i \neq 0$ .

### 3. RESTRICTION TO LEVI SUBGROUPS

For  $J \subseteq \Delta$ , let  $L_J$  be the Levi subgroup of  $G$  generated by the maximal torus  $T$  and the root subgroups corresponding to roots that are linear combinations of elements of  $J$ . Set

$$X_J(T)_+ := \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in J\}.$$

For  $\lambda \in X_J(T)_+$ , we can construct an induced module  $H_J^0(\lambda) := \text{ind}_{L_J \cap B}^{L_J} \lambda$  with simple  $L_J$ -socle  $L_J(\lambda)$ , and dually a Weyl module  $V_J(\lambda)$  with head  $L_J(\lambda)$ .

**Theorem 3.1.** *Let  $G$  be a simple simply connected algebraic group and  $J \subseteq \Delta$ . If  $V(\lambda) \otimes k$  is an irreducible  $G$ -module, then  $V_J(\lambda) \otimes k$  is an irreducible  $L_J$ -module.*

*Proof.* For  $k$  of characteristic 0,  $V_J(\lambda)$  is just the set of fixed points of  $Q_J$  on  $V(\lambda)$  (the unipotent radical of the parabolic  $P_J = L_J Q_J$ ); this is part of [Smi82]. Taking a  $\mathbb{Z}$ -form and reducing modulo  $p$ , we see that the dimension of the space of  $Q_J$  on  $V(\lambda)$  can only go up in characteristic  $p$ .

So if  $V(\lambda) = L(\lambda)$ , then again by [Smi82], the fixed points of  $Q_J$  on this module is  $L_J(\lambda)$  but has dimension at least  $V_J(\lambda)$ . The other inequality is clear since  $L_J(\lambda)$  is a quotient of  $V_J(\lambda)$ , so  $L_J(\lambda) = V_J(\lambda)$ .  $\square$

**Remark 3.2.** Given a group  $G$  and a particular prime  $p$ , there are few known necessary and sufficient condition in terms of  $\lambda$  for the Weyl module  $V(\lambda) \otimes k$  to be irreducible over every field  $k$  of characteristic  $p$ . There is an easy-to-apply statement for  $G = \text{SL}_2$ . For  $G = \text{SL}_n$ , Jantzen gives a necessary and sufficient condition, but it is less easy to

apply. There are also sporadic results in one direction or another, such as consequences of the Linkage Principle like [Jan03, II.6.24] or irreducibility when  $\lambda$  is restricted and  $\dim V(\lambda)$  is small. Theorem 3.1 provides an easy way to get necessary conditions on  $\lambda$  by taking various small  $J$ . Writing  $\lambda = \sum c_i \omega_i$  and taking  $J = \{\alpha_i\}$  one can apply the  $\mathrm{SL}_2$  criterion to constrain the possible values of  $c_i$ . Taking  $J$  to be pairs of adjacent roots of the same length allows one to reduce to the case of  $A_2$ , for which a lot is known, see [Jan03, II.8.20].

We mention the following related result that includes the case where  $V(\lambda) \otimes k$  is reducible.

**Proposition 3.3.** *For every  $\lambda \in X(T)_+$ , every  $J \subseteq \Delta$ , and every field  $k$ , the irreducible representation  $L_J(\lambda)$  of  $L_J$  is a direct summand of  $L(\lambda)|_{L_J}$ .*

*Proof.* For the sake of completeness we describe the analysis given in [CN11, Section 8] which follows [Smi82] and [Jan03, II.5.21]. There exists a weight space decomposition for the induced module given by

$$H^0(\lambda) = \left( \bigoplus_{\nu \in \mathbb{Z}J} H^0(\lambda)_{\lambda-\nu} \right) \oplus M.$$

where  $M$  is the direct sum of all weight spaces  $H^0(\lambda)_\sigma$  where  $\sigma \neq \lambda - \nu$  for any  $\nu \in \mathbb{Z}J$ . Furthermore,  $H_J^0(\lambda) = \bigoplus_{\nu \in \mathbb{Z}J} H^0(\lambda)_{\lambda-\nu}$  with the aforementioned decomposition being  $L_J$ -stable. This allows us to identify an  $L_J$ -direct summand

$$H^0(\lambda)|_{L_J} \cong H_J^0(\lambda) \oplus M. \quad (2)$$

By definition  $L(\lambda) = \mathrm{soc}_G(H^0(\lambda))$ . This implies that  $\mathrm{soc}_{L_J} L(\lambda) \subseteq \mathrm{soc}_{L_J}(H^0(\lambda))$ . Note that

$$L_J(\lambda) = \mathrm{soc}_{L_J}(H_J^0(\lambda)) \subseteq \mathrm{soc}_{L_J}(H^0(\lambda)). \quad (3)$$

Now  $L_J(\lambda)$  appears as an  $L_J$ -composition factor of  $L(\lambda)$  and  $H^0(\lambda)$  with multiplicity one. Consequently,  $L_J(\lambda)$  must occur in  $\mathrm{soc}_{L_J} L(\lambda)$ .

One can also apply the same argument for Weyl modules and see that

$$V(\lambda)|_{L_J} \cong V_J(\lambda) \oplus M'. \quad (4)$$

for some  $L_J$ -module  $M'$ . By an argument dual to the one in the preceding paragraph, we deduce that  $L_J(\lambda)$  appears in the head of  $L(\lambda)|_{L_J}$ . The fact that  $L_J(\lambda)$  has multiplicity one in  $L(\lambda)$  now shows that  $L_J(\lambda)$  is an  $L_J$ -direct summand of  $L(\lambda)$ .  $\square$

#### 4. THE CASE OF FUNDAMENTAL WEIGHTS

We now verify Theorem 1.1 for every fundamental weight. We abuse notation and write  $V(\lambda)$  instead of  $V(\lambda) \otimes k$ .

**Type  $A_n$  ( $n \geq 1$ ).** In this case, all the fundamental weights are minuscule, so  $V(\lambda) = L(\lambda)$  for all  $\lambda = \omega_j$ ,  $j = 1, 2, \dots, n$ .

**Type  $B_n$  ( $n \geq 2$ ).** For  $B_n$ , the fundamental weight  $\omega_n$  is minuscule. We claim that  $V(\omega_i)$  is reducible for  $1 \leq i < n$  and  $\text{char } k = 2$ .

The split adjoint group of type  $B_n$  is  $\text{SO}(q)$  for a quadratic form  $q$  on a vector space  $X$  of dimension  $2n + 1$  where the tautological action on  $X$  is  $V(\omega_1)$ , see [KMRT98] or [Bor91]. As  $\text{char } k = 2$ , the bilinear form  $b_q$  deduced from  $q$  by the formula  $b_q(x, y) := q(x + y) - q(x) - q(y)$  is necessarily degenerate with 1-dimensional radical, providing an  $\text{SO}(q)$ -invariant line, call it  $S$ .

For  $2 \leq i < n$ , we restrict to the Levi subgroup of type  $B_{n-i+1}$  corresponding to  $J = \{\alpha_i, \alpha_{i+1}, \dots, \alpha_n\}$ . By the previous paragraph,  $V_J(\omega_i)$  is reducible in characteristic 2, hence  $V(\omega_i)$  is by Theorem 3.1.

Alternatively, one can see the reducibility concretely by noticing that  $V(\omega_i)$  has the same character and dimension as  $\Lambda^i(V(\omega_1))$ , because this is so in case  $k = \mathbb{C}$ . In particular,  $\Lambda^i(V(\omega_1))$  has a unique maximal weight, the highest weight of  $V(\omega_i)$ , and there is a nonzero  $\text{SO}(q)$ -equivariant map  $\phi: V(\omega_i) \rightarrow \Lambda^i(V(\omega_1))$ . As  $S \wedge \Lambda^{i-1}(V(\omega_1))$  is a proper and  $\text{SO}(q)$ -invariant subspace of  $\Lambda^i(V(\omega_1))$ , it follows that  $V(\omega_i)$  is reducible.

**Type  $D_n$  ( $n \geq 4$ ).** For type  $D_n$ , the fundamental weights  $\omega_1, \omega_{n-1}$ , and  $\omega_n$  are minuscule. We claim that  $V(\omega_i)$  is reducible for  $2 \leq i \leq n - 2$  and  $\text{char } k = 2$ .

The representation  $V(\omega_2)$  has the same character and dimension as  $\Lambda^2(V(\omega_1))$ . The alternating bilinear form  $b_q$  deduced from the invariant quadratic form  $q$  on  $V(\omega_1)$  gives an invariant line in  $\Lambda^2(V(\omega_1))$  — i.e.,  $D_n$  maps into  $C_n$ , which is already reducible on  $\Lambda^2(V(\omega_1))$  — proving the claim for  $i = 2$ .

Alternatively,  $V(\omega_2)$  is the adjoint action on the Lie algebra of  $\text{Spin}_{2n}$  (when  $\text{char } k = 2$ , this is distinct from  $\text{Lie}(\text{SO}_{2n})$ ), and the center  $S$  is a proper submodule, namely  $\text{Lie}(\mu_2 \times \mu_2)$  (if  $n$  is even) or  $\text{Lie}(\mu_4)$  (if  $n$  is odd).

For  $2 < i \leq n - 2$ , we may use either of the arguments employed in the  $B_n$  case: restriction to the Levi subgroup of type  $D_{n-i+2}$  corresponding to  $J = \{\alpha_{i-1}, \alpha_i, \dots, \alpha_n\}$ , where we find that  $V_J(\omega_i)$  is reducible by the previous paragraph, or by noting that there is an equivariant map  $V(\omega_i) \rightarrow \Lambda^i(V(\omega_1))$  and that  $S \wedge \Lambda^{i-2}(V(\omega_1))$  is a proper submodule in the codomain.

**Type  $C_n$  ( $n \geq 3$ ).** For type  $C_n$  with  $n \geq 3$ , the fundamental weight  $\omega_1$  is minuscule. For  $\omega_2$ , [PS83, Th. 2(iv)] gives that  $V(\omega_2)$  is reducible when  $\text{char } k = p$  if and only if the prime  $p$  divides  $n$ , compare [Jan03, p. 287]. For  $\omega_i$  with  $2 < i < n$ , restricting to the Levi of type  $C_{n-i+2}$  corresponding to  $J = \{\alpha_{i-1}, \alpha_i, \dots, \alpha_n\}$  shows that  $V_J(\omega_i)$  is reducible if  $p$  divides  $n - i + 2$ . For  $i = n$ , we restrict to the Levi subgroup of type  $C_2 = B_2$  corresponding to  $J = \{\alpha_{n-1}, \alpha_n\}$  to find that  $V_J(\omega_n)$  is the 5-dimensional natural module for  $B_2$ , which is reducible in characteristic 2.

**Exceptional types.** For exceptional types, tables of which fundamental weights  $\omega$  have  $V(\omega)$  reducible in which characteristics can be found in [Jan91, p. 299] or, for smaller dimensions, in [Lüb01]. These confirm our main theorem, and, in case  $V(\omega) \otimes k$  is reducible for some  $k$ , it is so for a  $k$  with  $\text{char } k = 2$  or 3.

We remark that for the representations  $V(\omega_i)$  of  $E_8$  for  $i \neq 8$ , one can verify Theorem 1.1 by restricting to a Levi and using induction, instead of referring to [Jan91] or [Lüb01] directly.



Because it is such an important example, we mention specifically that the adjoint representation  $V(\omega_8)$  of  $E_8$  is irreducible because  $\text{Lie}(E_8)$  is simple for every field, see [Ste61] or [Hog82]. Here is an alternative argument provided to us by Gross: As  $E_8$  is simply-laced, the Weyl group acts transitively on the roots, so the normalizer  $N_{E_8}(T)$  of a split torus has an irreducible submodule in the adjoint representation  $\text{Lie}(E_8)$  given by the sum of all the root spaces. The miracle that is special to  $E_8$  is that the Weyl group acts irreducibly on the submodule  $\text{Lie}(T)$ , which is the  $E_8$ -lattice mod  $\text{char } k$ .<sup>2</sup> Then the restriction of the representation  $\text{Lie}(E_8)$  to  $N_{E_8}(T)$  is the direct sum of two irreducible representations, one of dimension 240 and the other of dimension 8. Since  $E_8$  has no nontrivial map into  $\text{SL}_8$ , it does not preserve either submodule, and so acts irreducibly on  $\text{Lie}(E_8)$ .

This is in contrast to the case where  $G$  is simple of type other than  $E_8$ , where  $N_G(T)$  acts reducibly on  $\text{Lie}(T)$  for some characteristic (2 for types  $B, C, D, E_7$  and  $F_4$ ; 3 for  $E_6$  and  $G_2$ ; and dividing  $n$  for type  $A_{n-1}$ ). And of course if  $G$  has roots of different lengths and is simply connected, then for  $\text{char } k = 2$  or 3, the short roots generate a subalgebra of  $\text{Lie}(G)$  invariant under  $G$ , see e.g. [His84, Lemma 3.2] or [Ste61, p. 1121].

Here is yet another argument to see that  $\text{Lie}(E_8)$  is an irreducible representation for every field  $k$ . Namely, it is a special case of the following observation: *If  $G$  is simple and simply connected over a field  $k$ , the center  $Z$  of  $G$  is étale<sup>3</sup>, and all the roots of  $G$  have the same length, then  $\text{Lie}(G)$  is an irreducible representation of  $G$ .* To prove this general statement, note that the natural map  $\text{Lie}(G) \rightarrow \text{Lie}(G/Z)$  has kernel  $\text{Lie}(Z) = 0$ , so is an isomorphism by dimension count. But the domain is the Weyl module  $V(\tilde{\alpha})$  and the codomain is its dual  $V(\tilde{\alpha})^* = H^0(\tilde{\alpha})$  because of the assumption on the roots [Gar09, 3.5]. Since  $V(\tilde{\alpha}) \cong H^0(\tilde{\alpha})$ , they are irreducible  $G$ -modules.

## 5. TYPE $B_n$ , WEIGHT $\omega_1 + \omega_n$

Let  $k$  be an algebraically closed field of characteristic  $p \geq 0$ . Let  $G = \text{Spin}_{2n+1}(k)$  for  $n \geq 2$ . The irreducible  $G$ -module  $L(\omega_1)$  has dimension  $2n + 1$  if  $\text{char } k \neq 2$  and dimension  $2n$  if  $\text{char } k = 2$ . Moreover, the irreducible  $G$ -module  $L(\omega_n)$  is the spin module for  $G$  of dimension  $2^n$ . In this section we show the following, which amounts to a specific case of Theorem 1.1.

**Theorem 5.1.** *Let  $G = \text{Spin}_{2n+1}(k)$  with  $n \geq 2$ . Then*

$$\dim L(\omega_1 + \omega_n) = \begin{cases} 2^n \cdot 2n & \text{if } \text{char } k \text{ does not divide } 2n + 1; \\ 2^n \cdot (2n - 1) & \text{if } \text{char } k \text{ does divide } 2n + 1. \end{cases}$$

The proof will appear at the end of the section. The analysis will entail the restriction of modules to a monomial subgroup of  $\text{SO}_{2n+1}$  via its lift to  $\text{Spin}_{2n+1}$  and the use of permutation modules for the alternating group.

Let  $U := L(\omega_1) \otimes L(\omega_n)$ . If  $p = 0$  (and so also for all but finitely many  $p$ ), this is a direct sum of two composition factors  $L(\omega_1 + \omega_n)$  and  $L(\omega_n)$ . In particular, the Weyl module for the dominant weight  $\omega_1 + \omega_n$  has dimension  $2n \cdot 2^n$ .

<sup>2</sup>This is an illustration of a specific case, for  $G$  the Weyl group of  $E_8$ , of Thompson's question mentioned in the introduction.

<sup>3</sup>For example, this is true if  $\text{char } k$  is “very good” for  $G$ .

If  $p = 2$  then as in [Ste63],  $U$  is  $L(\omega_1 + \omega_n)$ , verifying the theorem. We assume for the rest of the section that  $p$  is odd.

Note that in  $G/Z(G) = \mathrm{SO}_{2n+1}(k)$ , there is a finite subgroup  $X$  isomorphic to  $A.A_{2n+1}$  where  $A$  is an elementary abelian 2-group of rank  $2n$  and  $A_{2n+1}$  denotes the alternating group on  $2n+1$  symbols. The group  $X$  is the derived subgroup of the group of orthogonal transformations preserving an orthogonal set of  $2n+1$  lines.

Let  $H$  denote the lift of  $X$  to  $G$ . Let  $E$  be the lift of  $A$  to  $G$ . First we note:

**Lemma 5.2.**  *$E$  is extraspecial of order  $2^{1+2n}$ .*

*Proof.* Since  $X$  acts irreducibly on  $E/Z(G)$ ,  $E$  is either elementary abelian or extraspecial of the given order. By induction it suffices to see that  $E$  is nonabelian in the case  $n = 2$  (actually we could start with  $n = 1$ ). This is clear since  $\mathrm{Spin}_2(k) \cong \mathrm{Sp}_4(k)$  and so contains no rank 5 elementary abelian 2-groups.  $\square$

Note that  $H/E \cong A_{2n+1}$ . Let  $H_1$  be a subgroup of  $H$  containing  $E$  with  $H_1/E \cong A_{2n}$ .

The group  $E$  has a unique faithful irreducible module over  $k$  of dimension  $2^n$  that is the restriction of  $L(\omega_n)$ . (It is a tensor product of  $n$  2-dimensional representations of the central factors of  $E$ , cf. [Gor80, 5.5.4, 5.5.5].) Since  $Z(G) = Z(E)$  acts nontrivially on  $U$ , every composition factor for  $E$  on  $U$  is isomorphic to  $L(\omega_n)$ . It follows immediately that  $L(\omega_1)$  and  $L(\omega_n)$  are each irreducible for  $H$ . Note also that  $L(\omega_1)$  is induced from a linear character  $\phi$  of  $H_1$ . Thus, as an  $H$ -module,  $U \cong L(\omega_n) \otimes \phi_{H_1}^H$ . In fact, we see that we can replace  $\phi$  by the trivial character of  $H_1$ :

**Lemma 5.3.**  *$U \cong L(\omega_n) \otimes k_{H_1}^H$  as an  $H$ -module.*

*Proof.* It suffices to show that  $L(\omega_n) \otimes \phi_{H_1} \cong L(\omega_n) \otimes k_{H_1}$  as  $H_1$ -modules. Note that they are both irreducible since they are irreducible  $E$ -modules. If  $n = 2$ , the result is easy to see (alternatively, one can modify the argument below). So assume that  $n > 2$ .

In fact, we observe that any  $H_1$ -module  $V_1$  that is isomorphic to  $L(\omega_n)$  as an  $E$ -module is isomorphic to  $L(\omega_n)$  as an  $H_1$ -module. This follows by noting that  $\mathrm{Hom}_E(L(\omega_n), V_1)$  is 1-dimensional and since  $H_1/E$  is perfect,  $H_1/E$  acts trivially on this 1-dimensional space, whence  $\mathrm{Hom}_{H_1}(L(\omega_n), V_1)$  is also 1-dimensional. Since the two modules are irreducible, this shows they are isomorphic.  $\square$

**Lemma 5.4.**  $\dim \mathrm{Hom}_H(U, U) = 2$ .

*Proof.* This follows by Lemma 5.3 and Frobenius reciprocity.  $\square$

Let  $V$  be the unique nontrivial composition factor of  $k_{A_{2n+1}}^{A_{2n+1}}$  (for  $n > 1$ ). This has dimension  $2n$  if  $p$  does not divide  $2n+1$  and dimension  $2n-1$  if  $p$  does divide  $2n+1$ .

By [Dad80] or [Nav98, Cor. 8.19], we know:

**Lemma 5.5.** *Viewing  $V$  as an  $H$ -module (that is trivial on  $E$ ),  $L(\omega_n) \otimes_k V$  is irreducible.*

$\square$

*Proof of Theorem 5.1.* From the above, we see that  $U$  has two  $H$ -composition factors if  $p$  does not divide  $2n+1$  and three composition factors if  $p$  does divide  $2n+1$ . This immediately implies that if  $p$  does not divide  $2n+1$ , then  $L(\omega_1 + \omega_n)$  is irreducible for  $H$  and has dimension  $2n \cdot 2^n$  (whence also for  $G$ ).



Now assume that  $p$  does divide  $2n + 1$ . For sake of contradiction, suppose that  $L(\omega_1 + \omega_n)$  has the same dimension as the Weyl module  $V(\omega_1 + \omega_n)$  for  $G$ , so  $U$  has precisely two nonisomorphic composition factors as a  $G$ -module,  $L(\omega_1 + \omega_n)$  and  $L(\omega_n)$ . Since  $U$  is self-dual it would be a direct sum of the two modules.

Recall  $U$  has three  $H$ -composition factors (two isomorphic to  $L(\omega_n)$ ). Thus, the  $G$ -submodule  $L(\omega_1 + \omega_n)$  must have two nonisomorphic  $H$ -composition factors. Again, since  $L(\omega_1 + \omega_n)$  is self dual, this implies that  $U$  is a direct sum of three simple  $H$ -modules. This contradicts Lemma 5.4 and completes the proof of Theorem 5.1.  $\square$

Our analysis shows that when  $p \mid 2n + 1$  the Weyl module  $V(\omega_1 + \omega_n)$  has two composition factors:  $L(\omega_1 + \omega_n)$ ,  $L(\omega_n)$ . Therefore, one can apply [Jan03, II.2.14] to determine  $\text{Ext}^1$  between these simple modules.

**Corollary 5.6.** *Let  $G = \text{Spin}_{2n+1}$ . Then*

$$\dim \text{Ext}_G^1(L(\omega_1 + \omega_n), L(\omega_n)) = \begin{cases} 0 & \text{if char } k \text{ does not divide } 2n + 1; \\ 1 & \text{if char } k \text{ does divide } 2n + 1. \end{cases}$$

*Remark.* Alternatively, one might approach proving Theorem 5.1 by using the Sum Formula [Jan03, p. 283] or by adapting the arguments in [And80]. Either way, in the case where  $p$  divides  $2n + 1$ , one has to check a fact about the root system such as: for each positive root  $\alpha \neq \omega_1$ , each natural number  $m$  such that  $s_{\alpha, mp} \cdot (\omega_1 + \omega_n) < \omega_1 + \omega_n$ , and  $w$  in the Weyl group such that  $w(s_{\alpha, mp} \cdot (\omega_1 + \omega_n))$  is dominant,  $w \cdot (s_{\alpha, mp} \cdot (\omega_1 + \omega_n))$  is not dominant.

## 6. PROOF OF THEOREM 1.1

We now prove Theorem 1.1 by induction on the Lie rank of  $G$ .

**Type  $A_1$ .** In case of rank 1,  $G$  is  $\text{SL}_2$  and  $V(d) \otimes k$  is irreducible if and only if, for  $p = \text{char } k$ ,  $d + 1 = cp^e$  for some  $0 < c < p$  and  $e \geq 0$  [Win77, pp. 239, 240]. (This can be seen by comparing dimensions: Write out  $d$  in base  $p$  as  $d = \sum_i c_i p^i$ . Then  $\dim V(d) = d + 1$  whereas the irreducible module  $L(d)$  over  $k$  has dimension  $\prod_i (c_i + 1)$  by Steinberg's tensor product theorem.) As a consequence, for  $d \geq 2$ , it is impossible for  $V(d) \otimes \mathbb{F}_p$  to be irreducible for both  $p = 2$  and  $3$ .

An alternate argument (as noted by Andersen) can be provided if one does not require  $p \leq 3$ . Choose a prime  $p$  dividing  $d$ . Now  $\dim V(d) = d + 1$  with  $V(d) \twoheadrightarrow L(d) \cong L(d/p)^{(1)}$ . But,  $\dim L(d/p)^{(1)} \leq \frac{d}{p} + 1 < d + 1 = \dim V(d)$ . So  $V(d)$  is reducible in characteristic  $p$ .

**Reductions.** So suppose  $\text{rank } G \geq 2$  and Theorem 1.1 holds for all groups of lower rank.

Write  $\lambda = \sum c_i \omega_i$  with every  $c_i \geq 0$ . If some  $c_i > 1$ , then taking  $J = \{\alpha_i\}$ , the Levi subgroup  $L_J$  has semisimple type  $A_1$  and the restriction of  $V_J(\lambda)$  to  $L_J$  is reducible when  $\text{char } k$  is 2 or 3 by the argument for type  $A_1$ . Therefore, by Theorem 3.1 we may assume that  $c_i \in \{0, 1\}$  for all  $i$ .

If  $\lambda = 0$  or  $\lambda = \omega_i$  for some  $i$ , then we are done by §4. Hence, we may assume that at least two of the  $c_i$ 's are nonzero.

If there is a connected and proper subset  $J$  of  $\Delta$  such that  $c_i \neq 0$  for at least two indexes  $i$  with  $\alpha_i \in J$ , then we are done by induction and Theorem 3.1.

**Sums of extreme weights.** The remaining case is when the Dynkin diagram has no branches (i.e.,  $G$  has type  $A$ ,  $B$ ,  $C$ ,  $F_4$ , or  $G_2$ ) and  $\lambda = \omega_1 + \omega_n$  is the sum of dominant weights corresponding to the simple roots at the two ends of the diagram. For type  $A_n$ ,  $G = \mathrm{SL}_{n+1}$  and  $V(\omega_1 + \omega_n)$  is the natural action on  $\mathrm{Lie}(\mathrm{SL}_{n+1})$ , the trace zero matrices. If  $p$  divides  $n + 1$ , then the scalar matrices are a  $G$ -invariant subspace. Type  $B$  was handled in Theorem 5.1.

For type  $C_n$  with  $n \geq 3$ , we restrict to the Levi subgroup of type  $C_2$  and find that  $V_J(\omega_1 + \omega_n)$  has dimension 5 and is reducible in characteristic 2. Alternatively, as in [Ste63, §11], in characteristic 2 one finds  $L(\omega_1 + \omega_n) \cong L(\omega_1) \otimes L(\omega_n)$ , which has dimension  $n2^{n+1}$ , whereas by the Weyl dimension formula,

$$\dim V(\omega_1 + \omega_n) = 7 \cdot 2^{n-1} \cdot \frac{n(2n+1)}{n+3} \cdot \prod_{i=6}^{n+1} \frac{2i-3}{i} \quad \text{for } n \geq 4.$$

In the case of exceptional groups, for type  $F_4$ ,  $V(\omega_1 + \omega_4)$  is reducible in characteristic 2 because it has dimension 1053, yet by [Ste63]  $L(\omega_1 + \omega_4) \cong L(\omega_1) \otimes L(\omega_4)$  has dimension  $26^2 = 676$ . For type  $G_2$ ,  $\dim V(\omega_1 + \omega_2) = 64$  yet by Steinberg in characteristic 3  $L(\omega_1 + \omega_2)$  has dimension  $7^2 = 49$ . Alternatively, one can refer to [Lüb01, Tables A.49, A.50]. This completes the proof of Theorem 1.1.  $\square$

## 7. KILLING FORMS

The Weyl module  $V(\lambda)$  for a simply connected semisimple group  $G$  over  $\mathbb{Z}$  (or the Weyl module  $V(\lambda) \otimes k$  for a field  $k$ ) has a nonzero  $G$ -invariant bilinear form iff  $\lambda = -w_0\lambda$  for  $w_0$  the longest element of the Weyl group. Suppose this is the case; then up to sign there is a unique indivisible bilinear form  $b$  on  $V(\lambda)$ , and up to multiplication by a scalar  $b \otimes k$  is the unique  $G$ -invariant bilinear form on  $V(\lambda) \otimes k$ . Moreover,  $b \otimes k$  is nondegenerate iff  $V(\lambda) \otimes k$  is irreducible, because  $V(\lambda) \otimes k$  has a unique maximal proper submodule. Therefore, Theorem 1.1 gives:

**Corollary 7.1.** *Suppose  $\lambda = -w_0\lambda$  and  $G$  is simple, split, and simply connected. The following are equivalent:*

- (a) *The form  $b \otimes k$  on  $V(\lambda) \otimes k$  is nondegenerate for every field  $k$ .*
- (b) *The discriminant of  $b$  is  $\pm 1$ .*
- (c)  *$\lambda$  is minuscule or  $G = E_8$  and  $\lambda$  is the highest root  $\tilde{\alpha}$ .*

**The reduced Killing form.** Continue the notation from earlier in this section. The Killing form on  $\mathrm{Lie}(G) = V(\tilde{\alpha})$  is, up to sign,  $2h^\vee b$  for  $h^\vee$  the dual Coxeter number of  $G$  [GN04], and we choose the sign of  $b$  so that equality holds; it is natural to call  $b$  the *reduced Killing form* of  $G$ .<sup>4</sup> The discriminant of  $b$  in this case was calculated in [SS70, I.4.8].

Fix now an isogeny  $G \rightarrow \overline{G}$ . The isogeny gives an isomorphism  $\mathrm{Lie}(G) \otimes \mathbb{Q} \rightarrow \mathrm{Lie}(\overline{G}) \otimes \mathbb{Q}$  and so there is a unique minimal rational number  $e \geq 1$  so that  $eb$  is integer-valued and indivisible on  $\mathrm{Lie}(\overline{G})$ . We call  $eb$  and  $eb \otimes k$  the *reduced Killing form* on  $\mathrm{Lie}(\overline{G})$  and  $\mathrm{Lie}(\overline{G}) \otimes k$  respectively. Note that  $G$  and  $\overline{G}$  may have distinct Lie algebras;

<sup>4</sup>This normalization has the advantage, not enjoyed by the usual Killing form, that  $b \otimes k$  is nonzero for every  $k$ .

the natural map  $\mathrm{Lie}(G) \otimes k \rightarrow \mathrm{Lie}(\overline{G}) \otimes k$  has kernel the Lie algebra of the center of  $G \times k$ , which may be nonzero if the characteristic is not very good for  $G$ .

**Example 7.2.** Let  $G = \mathrm{SO}_{2n}$  for some  $n \geq 4$ , meaning the special orthogonal group of the quadratic form  $\sum_{i=1}^n x_i x_{n+1-i}$ . Then we can identify  $\mathfrak{so}_{2n} := \mathrm{Lie}(G)$ , as a  $\mathbb{Z}$ -algebra, with  $\{X - \sigma(X)\}$  where  $X$  is a  $2n$ -by- $2n$  matrix and  $\sigma(X) := SX^T S$  where  $S$  is the matrix with a diagonal of 1's running from the lower left corner to the upper right [KMRT98, p. 350]. A matrix  $X - \sigma(X)$  has characteristic polynomial a square and has trace divisible by 2; indeed there is a linear map  $\tau : \mathfrak{so}_{2n} \rightarrow \mathbb{Z}$  such that  $2\tau$  is the trace. The reduced Killing form on  $\mathfrak{so}_{2n}$  is  $(x, y) \mapsto \tau(xy)$ , cf. [Bou05, §VIII.13, Exercise 12].

Suppose now that  $\mathrm{char} k = 2$ . For  $n$  even,  $\mathfrak{so}_{2n} \otimes k$  is a uniserial  $\mathrm{SO}_{2n}$ -module with socle the scalar matrices, and codimension-1 subspace the kernel of  $\tau$ . For  $n$  odd,  $\mathfrak{so}_{2n} \otimes k$  is a direct sum of  $\ker \tau$  and the scalar matrices.

We prove the following, answering a question posed by George Lusztig. Recall that for  $n > 4$  even, the simply connected group  $\mathrm{Spin}_{2n}$  has two non-isomorphic quotients by a central  $\mu_2$ :  $\mathrm{SO}_{2n}$  and one more called a *half-spin group*; we denote it by  $\mathrm{HSpin}_{2n}$ .

**Theorem 7.3.** *Let  $\overline{G}$  be a simple split algebraic group over  $\mathbb{Z}$ . The reduced Killing form on  $\mathrm{Lie}(\overline{G}) \otimes k$  is nondegenerate for every field  $k$  if and only if  $\overline{G}$  is one of the following groups:*

- (a)  $E_8$ ;
- (b)  $\mathrm{SO}_{2n}$  for some  $n \geq 4$ ;
- (c)  $\mathrm{HSpin}_{2n}$  for  $n$  divisible by 4; or
- (d)  $\mathrm{SL}_{m^2} / \mu_m$  for some  $m > 1$ .

In the proof, we will use the following linear algebra exercise. Put  $M_r$  for the Cartan matrix of the root system of type  $A_r$ , which has 2's on the diagonal,  $-1$ 's on the super and subdiagonals, and 0's elsewhere. Then  $\det M_r = (r+1)$  and for every  $a, b$ ,

$$\det \left( \begin{array}{ccc|c} & & & a \\ & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline a & 0 & \cdots & 0 \\ & & & b \end{array} \right) = b(r+1) - a^2 r. \quad (5)$$

*Proof of Theorem 7.3.* By Corollary 7.1, we may assume that  $\overline{G}$  is not simply connected.

If  $\overline{G}$  is adjoint of type  $A$ ,  $D$ , or  $E$ , then  $\mathrm{Lie}(\overline{G})$  is the standard module  $H^0(\tilde{\alpha}) \cong V(\alpha)^*$ . By Theorem 1.1, there is some  $k$  for which  $V(\tilde{\alpha}) \otimes k$  is reducible, hence  $H^0(\tilde{\alpha}) \otimes k$  is reducible, hence the irreducible socle  $L(\tilde{\alpha})$  is proper and is contained in the radical of  $eb \otimes k$  on  $H^0(\tilde{\alpha}) \otimes k$  as in [GN16, Lemma 5.2].

If  $\overline{G}$  is adjoint of type  $B_n$  or  $C_n$ , suppose  $\mathrm{char} k = 2$ . The natural map  $\mathrm{Lie}(G) \otimes k \rightarrow \mathrm{Lie}(\overline{G}) \otimes k$  has kernel  $k$ ; put  $X$  for the codimension-1 image. If  $eb \otimes k$  is nondegenerate, then  $eb \otimes k|_X$  is nondegenerate or  $X$  contains a submodule isomorphic to  $k$ . But neither occurs. Indeed, the submodule structure of  $X$  is given in [His84, Hauptsatz]; it is uniserial with head of dimension  $2n$  and socle of dimension  $2n^2 - n - 1$  or  $2n^2 - n - 2$  (or with roles of head and socle reversed).

Therefore, we are reduced to considering the cases where  $\overline{G}$  is  $\mathrm{SO}_{2n}$  for some  $n \geq 4$ ;  $\mathrm{HSpin}_{2n}$  for some even  $n$ ; or  $\mathrm{SL}_n / \mu_m$  for some  $n \geq 4$  and  $m \neq 1, n$  dividing  $n$ . Fix a pinning for  $\overline{G}$ , which includes a choice of a maximal torus  $T$  and a generator  $x_\beta$  for the 1-dimensional root subalgebra corresponding to each root  $\beta$  of  $\overline{G}$  with respect to  $T$ .

Suppose  $e > 1$ . Then, as  $b(x_\beta, x_{-\beta}) \geq 1$  for all roots  $\beta$ ,  $eb$  vanishes on the subspace spanned by the  $x_\beta$ 's. That subspace has dimension greater than half of  $\dim \overline{G}$ , so  $eb$  is degenerate. Therefore,  $e = 1$ . In particular, if  $G = \mathrm{HSpin}_{2n}$ , then  $n$  is divisible by 4 by [Gar09, Example 3.7] and if  $G = \mathrm{SL}_n / \mu_m$  then  $m^2 \mid n$  by [Gar09, Lemma 5.2].

Furthermore,  $b \otimes k$  restricts to be nondegenerate on the subspace spanned by the  $x_\beta$ 's, so, if  $b \otimes k$  is nondegenerate for all  $k$ , the restriction of  $b$  to  $\mathrm{Lie}(T)$  has discriminant  $\pm 1$ ; we now calculate  $b$  for each of the remaining possibilities. Specifically,  $\mathrm{Lie}(T)$  is a lattice lying properly between the root lattice  $Q$  and the weight lattice  $P$ .<sup>5</sup> It is generated, as a lattice in  $P$ , by

- for  $\overline{G} = \mathrm{SO}_{2n}$ :  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \omega_1$ ;
- for  $\overline{G} = \mathrm{SL}_n / \mu_m$ :  $\alpha_2, \alpha_3, \dots, \alpha_n, \frac{n}{m}\omega_{n-1}$ ; and
- for  $\overline{G} = \mathrm{HSpin}_{2n}$ :  $\omega_1, \alpha_2, \alpha_3, \dots, \alpha_n$ ,<sup>6</sup>

see [Gar09, 3.6, 3.7, 5.2].

For  $\overline{G} = \mathrm{SO}_{2n}$ , the Gram matrix for  $b$  on  $\mathrm{Lie}(T)$  is as in (5) with  $r = n - 1$  and  $a = b = 1$ , so the matrix has determinant 1 as claimed in (b).

For  $\overline{G} = \mathrm{SL}_n / \mu_m$ , the matrix is also as in (5), with  $r = n - 2$ ,  $a = n/m$ , and  $b = n(n - 1)/m^2$ , so it has determinant  $n/m^2$ , which is 1 iff  $n = m^2$ .

For  $\overline{G} = \mathrm{HSpin}_{2n}$ , the Gram matrix has first row and first column  $(n/4, 0, \dots, 0, 1)$  and the remaining  $(n - 1)$ -by- $(n - 1)$  submatrix is the Cartan matrix of type  $D_{n-1}$ . This matrix has determinant 1 as in (c).  $\square$

*Remark.* The subdivision of groups of type  $D_n$  for  $n$  even into the cases  $n \equiv 0, 2 \pmod{4}$ , as seen in Theorem 7.3(c), can also be seen in the representation theory of these groups: the half-spin representation of  $\mathrm{Spin}_{2n}$  over  $\mathbb{C}$  is orthogonal for  $n$  divisible by 4 and symplectic for  $n \equiv 2 \pmod{4}$ , see [Bou05, Ch. VIII, Table 1] or [KMRT98, 8.4].

**Example 7.4.** Let  $\overline{G} := \mathrm{SL}_{p^3} / \mu_p$  for some prime  $p$ . Then the reduced Killing form on  $\mathrm{Lie}(\overline{G}) \otimes k$  is degenerate when  $\mathrm{char} k = p$ , yet  $\mathrm{Lie}(\overline{G}) \otimes k$  is self-dual for every field  $k$ . This is obvious if  $\mathrm{char} k \neq p$ , whereas if  $\mathrm{char} k = p$ ,  $\mathrm{Lie}(\overline{G}) \otimes k$  is a direct sum of the self-dual representations  $L(\tilde{\alpha})$  and  $k$  by [Hum67, 9.4].

## 8. COMPLEMENTS TO THEOREM 1.1

**Failure of converse to Theorem 3.1.** Our first complement to Theorem 1.1 is to make precise the settings where the converse to “ $V(\lambda)$  irreducible implies  $V_J(\lambda)$  irreducible” fails.

<sup>5</sup>Properly we should consider the root and weight lattice of the dual root system, but as  $\overline{G}$  has type  $A$  or  $D$ ,  $\Phi$  is canonically isomorphic to the dual system.

<sup>6</sup>There is a typo in [Gar09, Example 3.7].  $T_*$  is generated by  $Q$  and  $\omega_\ell$ , not  $\omega_1 + \omega_{\ell-1}$ . This correction does not change the conclusion of the example nor the rest of the paper.

**Theorem 8.1.** *Let  $\lambda$  be a dominant weight for  $G$  a split, simple, and simply connected group over  $\mathbb{Z}$  with  $\text{rank } G > 1$ . Then  $V_J(\lambda) \otimes k$  is irreducible for every  $J \subsetneq \Delta$  if and only if one of the following occurs:*

- (a)  $\lambda$  is minuscule.
- (b)  $\lambda$  is the highest short root  $\alpha_0$ .
- (c)  $\Phi = B_n$  and  $\lambda = \omega_1 + \omega_n$ .
- (d)  $\Phi = G_2$  and  $\lambda = \omega_2$  or  $\omega_1 + \omega_2$ .

*Proof.* First assume that one of conditions (a)–(d) holds. Then one can directly verify that  $V_J(\lambda) \otimes k$  is irreducible for every  $J \subsetneq \Delta$  (by Theorem 1.1).

On the other hand, suppose that  $V_J(\lambda) \otimes k$  is irreducible for every  $J \subsetneq \Delta$ . Write  $\lambda = \sum_i a_i \omega_i$ . If some  $a_i > 1$  or at least three of the  $a_i$  are nonzero, then by Theorem 1.1, we see that  $V_J(\lambda)$  is not irreducible with  $J$  obtained by removing an end node other than  $i$  in the first case or any end node in the second case.

Next consider the case when  $\lambda = \omega_i + \omega_j$ ,  $i \neq j$ . The result follows unless  $\{i, j\}$  correspond to all the end nodes. If there are three end nodes, this is not possible. Thus, we only need consider types  $A, B, C, F$  and  $G$ . If  $\Phi = A_n$ , this leads to (b). If  $\Phi = B_n$ , this leads to (c). Moreover, if  $\Phi = C_n$ ,  $n \geq 3$ , then  $V_J(\lambda)$  is not irreducible for  $J = \Delta - \{\alpha_1\}$ . Similarly, if  $\Phi = F_4$ , this case does not occur since  $V_J(\lambda)$  is not irreducible for  $J = \Delta - \{\alpha_4\}$ . If  $\Phi = G_2$ , this leads to one of the cases in (d).

It remains to consider the case that  $\lambda = \omega_i$  for some  $i$ . If  $\Phi = A_n$ , then  $\omega_i$  is minuscule. If  $G$  has rank 2, then removing a single node gives a Levi of type  $A_1$  and so we have irreducibility. So assume that  $\Phi$  is not of type  $A_n$  and has rank at least 3. It suffices to check that for any  $J$  obtained by removing an end node that  $V_J(\lambda)$  irreducible implies that  $\lambda$  is either minuscule or  $\lambda = \alpha_0$ .

Suppose that  $\Phi = D_n$ ,  $n \geq 4$ . If  $\lambda$  is not minuscule and  $\lambda \neq \alpha_0 = \omega_2$ , then we can remove the first node and see that  $V_J(\lambda)$  is not irreducible.

It remains to consider types  $B, C, E$  and  $F$ . If  $i$  does not correspond to an end node, then we can choose  $J$  in such a way that the Levi factor of the reduced system does not have type  $A_n$  and  $V_J(\lambda)$  does not correspond to an end node, whence by Theorem 1.1,  $V_J(\lambda)$  is not irreducible.

If  $G$  has type  $B_n$  or  $C_n$ , then  $\omega_1$  and  $\omega_n$  either correspond to the short root or are minuscule. In the case when  $G$  has type  $E_6$ , then  $\omega_i$  corresponding to an end node is either  $\alpha_0$  or minuscule. If  $G$  has type  $F_4$  or  $E_n$ ,  $n \geq 7$ , then one checks the only end node satisfying the hypotheses is  $\alpha_0$ .  $\square$

**Connection to  $B$ -cohomology.** Let  $B$  be the Borel of  $G$  corresponding to the negative roots. For  $2\rho$  the sum of the positive roots and  $N$  the number of positive roots, one can use Serre duality to show that

$$\text{Hom}_G(k, V(-w_0\lambda)) \cong \text{Ext}_B^N(k, \lambda - 2\rho) \cong H^N(B, \lambda - 2\rho), \quad (6)$$

see [HN06] and [GN16, Theorem 5.5].

For  $\lambda = \tilde{\alpha}$  the highest root,  $\tilde{\alpha} = -w_0\tilde{\alpha}$  and  $V(\tilde{\alpha})$  is the Lie algebra of the simply connected cover of  $G$ . The adjoint representation for  $E_8$  is simple for all  $p > 0$ , so

$$H^{120}(B, \tilde{\alpha} - 2\rho) = 0.$$



On the other hand, if  $G$  is of type  $A_n$  then

$$H^{n(n+1)/2}(B, \tilde{\alpha} - 2\rho) \cong \begin{cases} k & \text{if } p \mid n+1 \\ 0 & \text{if } p \nmid n+1. \end{cases}$$

Similar statements can be formulated for other types.

The calculation of the  $B$ -cohomology with coefficients in a one-dimensional representation is an open problem in general. Complete answers are known for degrees 0, 1, and 2 and for most primes in degree 3. See [BNP14] for a survey.

**Quantum Groups.** For quantum groups (Lusztig  $\mathcal{A}$ -form) at roots of unity, one can ask when the quantum Weyl modules are globally irreducible. The Weyl modules with minuscule highest weights will yield globally irreducible representations. One can prove an analog of Theorem 3.1 to use Levi factors to reduce to considering fundamental weights or weights of the form  $\omega_1 + \omega_n$ .

For type  $A_n$ , if the root of unity has order  $l$  and  $l \mid n+1$  then  $V(\omega_1 + \omega_n)$  is not simple (see [Fay05]). This uses representation theory of the Hecke algebra of type  $A_n$ . From this one can prove the analog of our main theorem (Theorem 1.1) for quantum groups in the  $A_n$  case.

In order to handle root systems other than  $A_n$ , more detailed information needs to be worked out such as the tables given in [Jan91] and analogs of results for Weyl modules in type  $C_n$  as given in [PS83].

**Further Directions.** Suppose now that  $G$  is a split simply connected algebraic group over  $\mathbb{Z}$  and  $\lambda$  is a dominant weight. In a preliminary version of this manuscript, we asked to what extent is the following statement true: *If  $\mu$  is a dominant weight that is maximal among the dominant weights  $< \lambda$ , then there is a field  $k$  such that  $V(\lambda) \otimes k$  has  $L(\mu)$  as a composition factor.* Certainly, it is false for  $G = E_8$ ,  $\lambda$  the highest root, and  $\mu = 0$ . Jantzen has recently shown that, apart from this one counterexample, the statement holds when  $G$  is simple. Note that, in contrast to Theorem 1.1, there is not an upper bound on  $\text{char } k$  that only depends on the rank of  $G$ . For example, take  $G = \text{SL}_2$ , a prime  $p$ , and  $d > p$  not divisible by  $p$ . Then  $d-2$  is a weight of  $L(d)$ , so  $L(d-2)$  is not in the composition series for  $V(d)$ .

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